PERSONAL TUTORIAL 5.

Analysis 01/2021

(1) **a.** (6 marks) Let a_n be a sequence of real numbers. Suppose that for any $n > m \geq 1$ we have $|\sum_{i=m}^{n} a_i| \leq \frac{1}{\sqrt{m}} - \frac{1}{\sqrt{n}}$. Prove that $\sum_{i=1}^{\infty} a_i$ converges.

b. (4 marks) Let a_n and b_n be sequences of non-zero real numbers and suppose that $\sum_{i=1}^{\infty} b_n$ is convergent and $\lim_{n\to\infty} |\frac{a_n}{b_n}| = 0$. Then either prove that $\sum_{i=1}^{\infty} a_n$ is convergent or find a counterexample.

c. (6 marks) Suppose that a_n is a monotonically increasing sequence and that a_n does not contain any subsequence that diverges to ∞. Then either prove that a_n is convergent or find a counter-example.

d. (4 marks) Let a_n be a real sequence that is not bounded above and is also not bounded below. Prove that a_n cannot have a convergent subsequence or find a counter-example.

LA&G 01/2021

(2) **a.** Let $V = \mathbb{R}^3$ be the $\mathbb{R}$-vector space with usual addition and scalar multiplication. Define the map:

$$\begin{aligned} T: \ V &\to V \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto \begin{pmatrix} c \\ 0 \\ b+c \end{pmatrix}. \end{aligned}$$

i. (3 marks) Show that T is a linear transformation.

ii. (2 marks) Find $Im\, T$ and $\ker T$.

iii. (4 marks) Show that $\left\{ T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $Im\, T$.

b. Now let $n \in \mathbb{N}$, and let W be and $\mathbb{R}$-vector space with basis $\{v_1, \ldots, v_{3n}\}$ and suppose that $T : W \to W$ is a linear transformation with:

$$T(v_i) = \begin{cases} 0_W & \text{if } i \equiv 0 \pmod 3 \\ v_{(i+1(\text{mod} 3n))} & \text{if } i \equiv 1 \pmod 3 \\ v_i + v_{(i+2(\text{mod} 3n))} & \text{if } i \equiv 2 \pmod 3 \end{cases}$$

i. (6 marks) Find $Im\, T$.

ii. (2 marks) Find a basis for $Im\, T$.

iii. (3 marks) Find the dimension of $\ker T$.

Calculus 01/2020

(3) **a.** (2 marks) Starting from the definition of $\cosh x = \frac{e^x+e^{-x}}{2}$, show that $\cosh 2x = 2\cosh^2 x - 1$.

b. (5 marks) By the use of the substitution $x = a\sinh\theta$, show that

$$\int (a^2 + x^2)^{1/2} dx = \frac{1}{2}x(a^2 + x^2)^{1/2} + \frac{1}{2}a^2 \sinh^{-1}(x/a) + K.$$

where K is a constant.

c. Two drones, labelled drone 1 and drone 2, set off from the origin $x = y = 0$ at the same instant and travel at constant speed u_1 and u_2, respectively. Drone 1 travels along the path $y = x$ and drone 2 travels along $y = 2\sqrt{x}$ ($x > 0$ for both paths).

i. (2 marks) Sketch the drone trajectories for both and find the point P where the trajectories cross.

ii. (3 marks) If the drones meet at P find the angle between their instantaneous velocities.

iii. (8 marks) The drone speeds are programmed so that they meet at P after a preset time. Show that for this to happen the speeds have to obey $\frac{u_2}{u_1} = \frac{2\sqrt{5}+\sinh^{-1} 2}{4\sqrt{2}}$. Why do we expect this ratio to be bigger than unity?

[*If you encounter integrals such as that in (b), simply use the result without re-calculation.*]

Probability 01/2021

(4) **a.** (3 marks) Consider a group of $n \in \mathbb{N}$ people, where n is even. Suppose there are 105 possible ways of splitting this group into pairs. Find n.

b. Suppose there are $n \in \mathbb{N}$ students in a class. Each student has exactly one pencil. All n pencils are distinct. The class teacher collects all pencils to check them before an exam and then hands them back to the students in a random order.

Let p_n denote the probability that at least one student gets their own pencil back.

i. (5 marks) Find p_n.
Hint: Set $A_i = \{\text{Student } i \text{ gets their own pencil back}\}$ and use the inclusion-exclusion principle.

ii. (1 mark) Find $\lim_{n\to\infty} p_n$.

c. Let $\Omega = [0,1]$, $\mathcal{F} = \{\emptyset, [0,1/2), [1/2,1], \Omega\}$. Which of the following are discrete random variables on $(\Omega, \mathcal{F})$? Please justify your answer.

i. (2 marks) For $\omega \in \Omega$, define

$$X(\omega) = \begin{cases} 1 & \text{if } 0 \le \omega < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \le \omega \le 1. \end{cases}$$

ii. (2 marks) For $\omega \in \Omega$, define

$$Y(\omega) = \begin{cases} 1 & \text{if } 0 \le \omega <\le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < \omega \le 1. \end{cases}$$

d. Let X be a continuous random variable with probability density given by

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the random variable $Y = g(X)$ where

$$Y = g(X) = \begin{cases} X & \text{if } 0 \le X \le 1/2, \\ \frac{1}{2} & \text{if } X > \frac{1}{2}. \end{cases}$$

i. (1 mark) Find the image of Y.

ii. (3 marks) Find the cumulative distribution function of Y.

iii. (3 marks) Sketch the cumulative distribution function. Is Y a continuous or discrete random variable or is it neither continuous nor discrete?