PERSONAL TUTORIAL 3.

1. Question Sheet 1

Qu 1. A journalist wants to write an article about the island of Knights and Knaves. All the n islanders are either Knights (that is, they **never** lie) or Knaves (that is, they **always** lie). Everyone on the island knows each other. The journalist interviews every islander and obtains the following answers

$$
\begin{aligned}
A_1 &: \text{there is at least 1 Knave on the island.} \\
A_2 &: \text{there are at least 2 Knaves on the island.} \\
\vdots & \qquad\qquad\qquad \vdots \\
A_{n-1} &: \text{there are at least } n-1 \text{ Knaves on the island.} \\
A_n &: \text{there are } n \text{ Knaves on the island.}
\end{aligned}
$$

Is it possible for the journalist to know if there are more Knights or Knaves on the island?

Qu 2. **Extra question if time remains** Let f be the function on the positive integers defined by

$$
\begin{cases}
f(1) = 0 \\
f(2n) = 2f(n) + 1 \\
f(2n+1) = f(2n) - 1.
\end{cases}
$$

Prove that the value of $f(m)$ is 0 for infinitely many m and that there is x such that $f(x) = 2022$.

2. Question Sheet 2

Qu 1. **(MATH40003 Linear Algebra and Groups)**
Let $V = \{a_0 + a_2x^2 + a_4x^4 + a_6x^6 : a_0, a_2, a_4, a_6 \in \mathbb{R}\}$, the vector space over $\mathbb{R}$ with standard addition and scalar multiplication.

(a) Show that V is a subspace of $\mathbb{R}[x]$ (the set of polynomials in x with co-efficients in $\mathbb{R}$).
(b) Find a basis for V. *Justify your answer fully i.e. prove the basis you find is in fact a basis.*

Qu 2. **(MATH40004 Calculus and Applications)**
Consider the function $f(x) = x^{-p} \log x$, $x > 0$, and $p > 0$ a constant.

(a) Find all zeros of $f(x)$ and also find $\max_{x>0} f(x)$.
(b) Sketch the graph of the function - make sure you identify the relevant points.
(c) Determine all values of p, such that $\int_0^1 f(x)dx$ exists.
(d) Determine all values of p, such that $\int_1^\infty f(x)dx$ exists.
(e) If $p = 1/2$, find α such that $\int_0^\alpha f(x)dx = 0$. If $p = 2$, find β such that $\int_\beta^\infty f(x)dx = 0$.

Qu 3. **(Extra question)**

(a) Simplify the expression

$$(a + b)^2 - 4ab,$$

where a and b are non-negative real numbers.
Hence, prove that the arithmetic mean of a and b, $\dfrac{a+b}{2}$, is greater than or equal to their geometric mean, $\sqrt{ab}$.

(b) Given that

$$16^{x+y^2} + 16^{x^2+y} = 1,$$

where x and y are real, find all possible values of x and y.
[Hint: find in terms of x and y a lower bound for $16^{x+y^2} + 16^{x^2+y}$, and express this bound as a power of 4.]