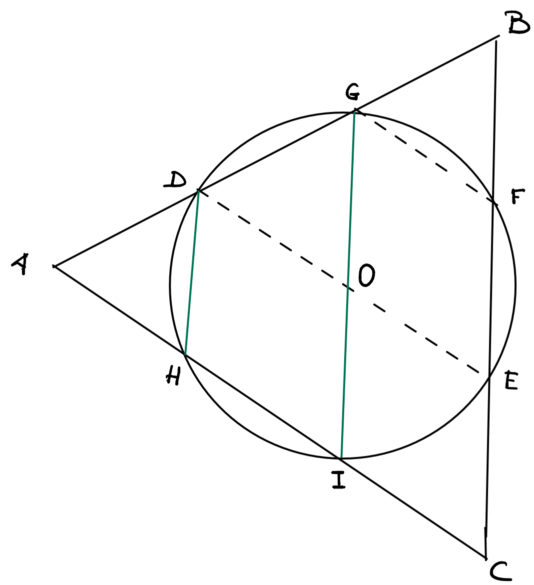
PERSONAL TUTORIAL 2.

1. Question sheet 1

Qu 1. Consider the following statement:

Proposition 1. *Let ABC be a triangle such that there is a circle passing through all the points dividing each side in 3 equal parts. Then ABC is equilateral*



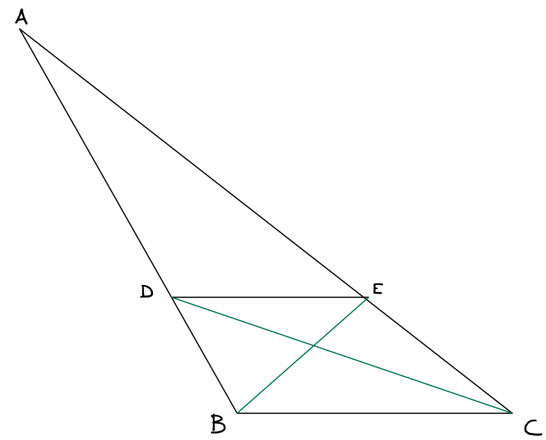
There are some errors in the proof below. Can you say where?

Proof. The triangles DOG, GOF and EOF have the same area. It follows that the segments DE and FG are parallel, so $DEFG$ is a trapezium. Since every trapezium that is inscribed in a circumference is isosceles, we have that DG and EF have the same length. The same argument holds for the segments DH and GI. We conclude that the third parts of each side are equal, which readily implies that the sides have equal length. □

Qu 2. The following proposition (also known as the intercept theorem) will fill the gap in the previous proof. The proof, recorded here in modern English, is taken form Euclid's Elements (Book VI, Proposition 2). This treatise was written more that 2300 years ago and is the oldest surviving text using the axiomatic method to prove mathematical statements. Read Euclid's proof, understand it and re-write it in your own words. (The proof is a re-interpretation of the translation in [1].)

Theorem 2. *If some straight line is drawn parallel to one of the sides of a triangle then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally then the straight line joining the cutting (points) will be parallel to the remaining side of the triangle.*

Proof. Let DE be parallel to one of the sides BC of the triangle ABC. I claim that BD is to DA as CE is to EA.



Indeed (the area of) triangle BDE is equal to (the area of) triangle CDE, because they are on the same base and between the same parallel lines [1, Prop. 1.38]. Consider now triangle ADE. Equal (magnitudes) have the same ratio to the same (magnitude) [1, Prop. 5.7]. Thus, triangle BDE is to triangle ADE, as triangle CDE is to triangle ADE. However, triangle BDE is to triangle ADE, as BD is to DA. For, having the same height –(namely), the (straight-line) drawn from E perpendicular to AB – they are to one another as their bases [1, Prop. 6.1]. Hence, for the same (reasons), triangle CDE is to ADE, as CE is to EA and, thus, BD is to DA, as CE is to EA [Prop. 5.11].

Let the sides AB and AC of triangle ABC have been cut proportionally (such that) BD is to DA, as CE is to EA. Let DE have been joined. I say that DE is parallel to BC.

By the same construction, BD is to DA as triangle BDE is to triangle ADE, and CE is to EA as triangle CDE is to triangle ADE [1, Prop. 6.1]. Thus, also, triangle BDE is to triangle ADE as triangle CDE is to triangle ADE [1, Prop. 5.11]. Thus, triangles BDE and CDE each have the same ratio to ADE. Thus, triangle BDE is equal to triangle CDE [1, Prop. 5.9], and they are on the same base DE. Equal triangles, which are also on the same base, are also between the same parallels [1, Prop. 1.39]. Thus, DE is parallel to BC.

□

Qu 3. Fill the gap in the proof of Proposition 1 using the intercept theorem.

References

[1] R. Fitzpatrick and J. Heiberg. (2007) Euclid's elements. [Online]. Available: `http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf`

2. Question sheet 2

Qu 1. 1. Prove the following Proposition using that $\sqrt{3}$ is not rational (which you also have to prove).

Proposition 3. *Let $m \in \mathbb{N}$, $m \neq 3$. Then $\sqrt{m} - \sqrt{3}$ is irrational.*

Lemma 4. *The real number $\sqrt{3}$ is irrational.*

2. Let p be a prime number. Prove the following Proposition using that $\sqrt{p}$ is irrational (which you also have to prove).

Proposition 5. *Let $m \in \mathbb{N}$, $m \neq p$. Then $\sqrt{m} - \sqrt{p}$ is irrational.*

3. (for everyone if there is extra-time) Show that if $m \in \mathbb{N}$ is not a square, then $\sqrt{m}$ is irrational.

Qu 2. 1. Consider the following two real matrices

$$A = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$$

Show that $A^2 - B^2 \neq (A+B)(A-B)$.
Let now

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Show that $N^2 - S^2 = (N+S)(N-S)$.

2. Let A, B be real square matrices. Find a necessary and sufficient condition on A and B such that $(A+B)(A-B) = A^2 - B^2$.
Let A be the real matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Determine all the matrices $B \in M_2(\mathbb{R})$ such that $(A+B)(A-B) = A^2 - B^2$.

Qu 3. 1. Show that every amount of postage greater than 1p can be formed using 2p and 3p stamps.

2. Show that there is $N \in \mathbb{N}$ such that all postage greater than N can be formed using only 2 and 5p stamps.